
DERIVING THE QUADRATIC FORMULA

This chapter assumes that you have finished solving quadratic equations by *Completing the Square*. You might understand at this point in your studies that the *factoring* method for solving quadratic equations seldom works, while Completing the Square *always* works.

And it works beautifully whether the solutions are integers, fractions (rational numbers), or radicals (irrational numbers). It even tells us when an equation has no solutions in $\mathbb{R}$, the system of real numbers

(which occurs when we reach the square root of a negative number).

But Completing the Square is quite boring and not the easiest thing in the world to do. There must be some way to complete the square on a “generic” quadratic equation, $ax^2 + bx + c = 0$, and end up with a formula that can be used to solve any quadratic equation more efficiently, whether it’s factorable or not.



"I FOUND OUT TODAY IN SCHOOL THAT I'M QUADRATIC EQUATION CHALLENGED!"

□ THE QUADRATIC FORMULA

We will start with a generic quadratic equation in standard form, and then try to *complete the square* . . . here we go:

$$ax^2 + bx + c = 0 \quad \text{[We assume } a > 0, \text{ because if it's not, we can multiply through by } -1 \text{ and make } a > 0.]$$

The technique of completing the square requires that we always have a leading coefficient of 1, and so the first step is to divide each side of the equation by a (which we know is not zero — otherwise, we wouldn't have a quadratic equation in the first place):

Divide each side of the equation by a :

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

To simplify the left side of the equation, just be sure you divide each term by a :

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a} \quad \text{(basically the reverse of adding fractions)}$$

And then simplify each side:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{(since } \frac{bx}{a} = \frac{b}{a} \cdot \frac{x}{1} = \frac{b}{a} \cdot x = \frac{b}{a}x \text{)}$$

Now bring the constant to the right side of the equation; that is, subtract $\frac{c}{a}$ from each side of the equation (we could have done this step first):

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

It's now time to compute the “magic number,” the quantity that will “complete the square.” We start with the coefficient of the linear term, in this case $\frac{b}{a}$, take half of it, and then square that result:

$$\frac{b}{a} \times \frac{1}{2} = \frac{b}{2a}, \text{ and then } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{(2a)^2} = \boxed{\frac{b^2}{4a^2}} \quad \leftarrow \text{The magic number}$$

It's time to add the magic number to each side of the equation:

$$x^2 + \frac{b}{a}x + \boxed{\frac{b^2}{4a^2}} = -\frac{c}{a} + \boxed{\frac{b^2}{4a^2}}$$

There are two things to do now: factor on the left and combine the fractions on the right. Let's factor first:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Notice that the variable x occurs just once in the entire equation (a good thing), buried amidst the parentheses on the left side of the equation.

Now we'll combine the fractions on the right side of the equation into a single fraction. Since the LCD of the denominators is $4a^2$, we need to multiply the top and the bottom of the first fraction by $4a$:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} \left[\frac{4a}{4a}\right] + \frac{b^2}{4a^2} \quad \left(\text{since } \frac{4a}{4a} = 1\right)$$

$$\text{or,} \quad \left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2} \quad (\text{We now have a common denominator})$$

Adding the fractions yields the equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

Using the commutative property for addition, we can reverse the two terms in the numerator of the right side of the equation:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

The next step we need in order to isolate the x is to take the square root of each side of the equation, remembering that every non-negative number has two square roots, denoted $\pm\sqrt{}$:

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad (2 \text{ square roots!})$$

To isolate the x (which is the whole point of this endeavor), we'll bring the term $\frac{b}{2a}$ to the right side of the equation (and put it in front):

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

We've isolated the x . Yay!
The rest is simplifying.

Now it's appropriate to split the radical into two separate radicals:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

Recall: $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ (for $y > 0$)

Simplify the radical in the denominator:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Technically speaking, $\sqrt{4a^2} = 2|a|$.
But since we assumed at the outset that $a > 0$, the radical becomes $2a$.

Finally, we combine the two fractions into a single fraction (can you believe the two fractions already have the same denominator? 😊):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In a single sentence: We've isolated the x in the quadratic equation $ax^2 + bx + c = 0$ by *completing the square*!

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are given by **The Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

□ NOTES ON THE QUADRATIC FORMULA

- The Quadratic Formula can be used to solve any quadratic equation, whether or not it's factorable, and whether its solutions are integers (whole numbers and their opposites), fractions, or even radicals (like $\sqrt{2}$). And if the quadratic equation has no solutions at all in the real numbers (that is, they have imaginary solutions), the Quadratic Formula will tell us that, too.
- Because of the $\pm$ sign, there are potentially two solutions. In fact, the two solutions of the quadratic equation $ax^2 + bx + c = 0$ can be written separately if you prefer:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- The Quadratic Formula is a single fraction:

$$\text{It is } \underline{\text{NOT}} \quad x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

If you don't write it as a single fraction at every stage of your solution, either you or your instructor will misread it and you'll be the one to lose points on the problem.

Homework

1. Which symbol in the Quadratic Formula is responsible for possibly giving us two solutions to a quadratic equation?
2. How many solutions of the quadratic equation will there be if the quantity inside the square root sign (called the *radicand*) is 100?
3. How many solutions of the quadratic equation will there be if the radicand is 30?

4. How many solutions of the quadratic equation will there be if the radicand is 0?
5. How many solutions of the quadratic equation will there be if the radicand is -9 ?
6. Consider the possibility that the value of a is 0 in the quadratic equation $ax^2 + bx + c = 0$. Answer the question, Does the Quadratic Formula apply, in two ways:
 - a. Consider what happens to the quadratic equation if $a = 0$.
 - b. Now analyze what happens if we actually allow $a = 0$ to wriggle its way into the Quadratic Formula.
7. Consider the quantity “ $b^2 - 4ac$,” the radicand inside the square root sign of the Quadratic Formula.
 - a. If $b^2 - 4ac > 0$, then the equation has ____ solution(s).
 - b. If $b^2 - 4ac = 0$, then the equation has ____ solution(s).
 - c. If $b^2 - 4ac < 0$, then the equation has ____ solution(s).

[The quantity $b^2 - 4ac$ is called the ***discriminant*** of the quadratic polynomial $ax^2 + bx + c$.]

Solutions

1. The plus/minus sign: $\pm$
2. Since $\sqrt{100} = 10$, and since there's a plus/minus sign in front of the radical, there will be two solutions.
3. Even though $\sqrt{30}$ doesn't result in a whole number, there's still a plus/minus sign in front of it, so there will also be two solutions.

4. Since $\sqrt{0} = 0$, and since ± 0 is just the single number 0, the plus/minus sign basically disappears, leaving one solution.
5. Since $\sqrt{-9}$ does not exist in early algebra classes, the Quadratic Formula has no meaning, and so there are no solutions.
6. First, if $a = 0$, the quadratic equation $ax^2 + bx + c = 0$ becomes $bx + c = 0$, which is NOT quadratic anymore. Indeed, we learned months ago how to solve this equation:

$$bx + c = 0 \Rightarrow bx = -c \Rightarrow x = -\frac{c}{b}$$

Second, if we ignore the above fact and use $a = 0$ in the Quadratic Formula, the denominator becomes $2a = 2(0) = 0$. That is, we're dividing by zero, which is undefined. However you look at it, the Quadratic Formula with $a = 0$ simply makes no sense.

7. a. 2 b. 1 c. 0 (assuming we want to stay within the real numbers.)

Postscript:

The derivation of the Quadratic Formula in this chapter was the classic method, using the method of completing the square. This was a fine technique to use, since completing the square will be used in your later studies of circles, parabolas, and other shapes — and the more practice, the better.

But there's another way to derive the Quadratic Formula, perhaps a way that may be easier for you to follow. Let's do it now — beginning, of course, with our quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

First, multiply each side of the equation by $4a$:

$$4a(ax^2 + bx + c) = 4a(0)$$

Distribute on the left and simplify on the right:

$$4a^2x^2 + 4abx + 4ac = 0$$

Bring the $4ac$ to the right side:

$$4a^2x^2 + 4abx = -4ac$$

Next, add b^2 to each side of the equation:

$$4a^2x^2 + 4abx + \boxed{b^2} = -4ac + \boxed{b^2}$$

Now factor the left side and flip the terms on the right side:

$$(2ax + b)^2 = b^2 - 4ac$$

Then remove the squaring by taking both square roots on the right:

$$2ax + b = \pm \sqrt{b^2 - 4ac}$$

Subtract b from each side of the equation:

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

And last, divide each side of the equation by $2a$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

WE DID IT!!

